

Some Instances of Homomesy Among Ideals of Posets

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Abstract

Given a permutation τ defined on a set of combinatorial objects S , together with some statistic $f : S \rightarrow \mathbb{R}$, we say that the triple $\langle S, \tau, f \rangle$ exhibits *homomesy* if f has the same average along all orbits of τ in S . This phenomenon was observed by Panyushev (2007) [4] and later studied, named and extended by Propp and Roby (2013) [7]. After Propp and Roby's paper, homomesy has received a lot of attention, and a number of mathematicians have been intrigued by it [8, 9, 10, 12, 13]. While seeming ubiquitous, homomesy is often surprisingly non-trivial to prove. Propp and Roby studied homomesy in the set of ideals in the product of two chains, with two well known permutations, rowmotion and promotion, the statistic being the size of the ideal. In this paper we extend their results to generalized rowmotion and promotion, together with a wider class of statistics in the product of two chains. Moreover, we derive similar results in other simply described posets. We believe that the framework we set up here can be fruitful in demonstrating homomesy results in ideals of broader classes of posets.

Consider a poset \mathcal{P} , and let $J(\mathcal{P})$ be the set containing all of the ideals in \mathcal{P} . The **rowmotion** operation, is an operation mapping $J(\mathcal{P})$ to itself, and it has been studied widely by combinatorists and under various names (Brouwer-Schrijver map [1], the Fon-der-Flaass map [6], the reverse map [4], and Panyushev complementation [3]). Rowmotion is defined as follows:

Definition 1. Given a poset \mathcal{P} on the elements of set \mathcal{S} , and an order ideal $I \in J(\mathcal{P})$, rowmotion is denoted by Φ^1 , and it is defined to be the down set ² of the minimal elements in $\mathcal{S} - I$.

Another interesting operation mapping $J(\mathcal{P})$ to itself is the **toggle map**. We can define the rowmotion operation also as the combination of several toggles. Toggling is defined as follows:

Definition 2. Given poset \mathcal{P} on the elements of set \mathcal{S} , an order ideal $I \in J(\mathcal{P})$, and an element $x \in \mathcal{S}$, the toggle map $\sigma_x : J(\mathcal{P}) \rightarrow J(\mathcal{P})$ is defined by:

¹Propp and Roby use Φ_J to denote rowmotion acting on order ideals and Φ_A for rowmotion acting on antichains. In this paper, we discuss only actions on order ideals, thus we drop the subscript.

²In a poset \mathcal{P} on elements of \mathcal{S} the down set of a set $\mathcal{X} \subseteq \mathcal{S}$ is the following: $\{y \in \mathcal{S} | \exists x \in \mathcal{X}, y \leq x\}$.

$$\sigma_x(I) = \begin{cases} I \cup \{x\}, & \text{if } x \notin I \text{ and } I \cup \{x\} \in J(\mathcal{P}). \\ I - \{x\}, & \text{if } x \in I \text{ and } I - \{x\} \in J(\mathcal{P}). \\ I, & \text{otherwise.} \end{cases} \quad (1)$$

Proposition 3. For all $x \in \mathcal{S}$ and $I \in J(\mathcal{P})$, $\sigma_x^2(I) = I$. If $x, y \in \mathcal{S}$ and x does not cover y nor y covers x , we have $\sigma_x \circ \sigma_y(I) = \sigma_y \circ \sigma_x(I)$.

We take a linear extension (x_1, \dots, x_n) of \mathcal{P} to be an indexing of the elements of \mathcal{P} that is $x_i < x_j$ in \mathcal{P} implies $i < j$. The following proposition was demonstrated in [2].

Proposition 4. [2] Given an arbitrary $I \in J(\mathcal{P})$ and linear extension (x_1, \dots, x_n) of \mathcal{P} , we have $\Phi(I) = \sigma_{x_1} \circ \sigma_{x_2} \circ \sigma_{x_3} \circ \dots \circ \sigma_{x_n}(I)$.

Definition 5. Let $\mathcal{Q}_{a,b} = [a] \times [b]$ ($[n] = \{1, 2, \dots, n\}$). Each element of the poset can be presented by a pair (i, j) , $i \in [a]$, $j \in [b]$ and $(i_1, j_1) \leq (i_2, j_2)$ iff $i_1 \leq i_2$ and $j_1 \leq j_2$.

In this paper, we are interested in the maps on $J(\mathcal{Q}_{a,b})$, as well as $J(\mathcal{U}_a)$ and $J(\mathcal{L}_a)$ where \mathcal{U}_a and \mathcal{L}_a are subsets of $\mathcal{Q}_{a,a}$ and defined as:

- $\mathcal{U}_a \subseteq \mathcal{Q}_{a,a}$, $\mathcal{U}_a = \{(i, j) | i, j \in [a], i \geq a+1-j\}$.
- $\mathcal{L}_a \subseteq \mathcal{Q}_{a,a}$, $\mathcal{L}_a = \{(i, j) | i, j \in [a], i \geq j\}$.

Notation. Let \mathcal{P} be one of $\mathcal{Q}_{a,b}$, \mathcal{U}_a or \mathcal{L}_a . By saying $(i, j) \in \mathcal{P}$ we are referring to the element in $[a] \times [b]$ with coordinates i and j . By saying $x = (i_1, j_1) \leq y = (i_2, j_2)$ we mean x is less than y in \mathcal{P} . To avoid confusion, we *never* use $(i, j) \in \mathcal{P}$ to indicate i is less than j in the partial order.

We call $\mathcal{Q}_{a,b}$ the square lattice or the product of two chains, \mathcal{U}_a the upper lattice and \mathcal{L}_a the left lattice. Among combinatorists \mathcal{U}_a is also known as the root poset of type A_a , and \mathcal{L}_a as the minuscule poset of type B_a or D_{a+1} . We employ the following terminology:

Definition 6. Let \mathcal{P} be one of $\mathcal{Q}_{a,b}$, \mathcal{U}_a or \mathcal{L}_a . For any arbitrary $I \in J(\mathcal{P})$,

We call the set of all points $(i, j) \in \mathcal{P}$ with constant $i+j$ a **rank**; $R_c(I) = \{(i, j) \in I | i+j = c\}$.

We call the set of all points $(i, j) \in \mathcal{P}$ with constant $i-j$ a **file**; $F_c(I) = \{(i, j) \in I | i-j = c\}$.

We call the sets of all points $(i, j) \in \mathcal{P}$ with constant i a **column**; $C_c(I) = \{(i, j) \in I | i = c\}$.

In the case when no ideal is specified we have $R_c = R_c(\mathcal{P})$, $F_c = F_c(\mathcal{P})$ and $C_c = C_c(\mathcal{P})$; \mathcal{P} should be clear from the context.

Example 7. The following figure shows $\mathcal{Q}_{5,4}$ and R_4 in it, \mathcal{U}_4 and F_{-1} in it and \mathcal{L}_4 and C_3 in it.

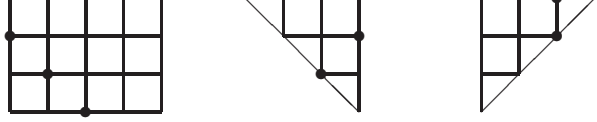


Figure 1. $\mathcal{Q}_{5,4}$ and $R_4(\mathcal{Q}_{5,4})$ \mathcal{U}_4 and $F_{-1}(\mathcal{U}_4)$ \mathcal{L}_4 and $C_3(\mathcal{L}_4)$

We can now define toggling for the above sets.

Definition 8. Consider the poset $\mathcal{Q}_{a,b}$ and $I \in J(\mathcal{Q}_{a,b})$. Let S be one of R_c or F_c for some arbitrary c . Letting $x_1 \dots x_m$ be some arbitrary indexing of the elements of S , we define $\sigma_S(I) = \sigma_{x_1} \circ \sigma_{x_2} \circ \dots \circ \sigma_{x_m}(I)$. Note that no two elements x_i, x_j of S constitute a covering pair, thus σ_S is well defined. For $S = C_c$, let $S = \{x_1, x_2, \dots, x_m\}$ where $x_1 < x_2 < \dots < x_m$. We define $\sigma_S(I) = \sigma_{x_1} \circ \sigma_{x_2} \circ \dots \circ \sigma_{x_m}(I)$.

Striker and Williams studied the class of so-called rc-posets, whose elements are partitioned into ranks and files³. Here, we will discuss the special rc-posets of the form $\mathcal{Q}_{a,b}$, \mathcal{U}_a or \mathcal{L}_a . The following definitions are from [5], restricted to the product posets of interest to us.

Definition 9. [5] Consider $\mathcal{Q}_{a,b}$. Let ν be a permutation of $\{2, \dots, a+b\}$. We define Φ_ν to be $\sigma_{R_{\nu(a+b-1)}} \circ \sigma_{R_{\nu(a+b-2)}} \circ \dots \circ \sigma_{R_{\nu(1)}}$.

Having Proposition, 4 it can be concluded that for $\nu = (a+b, a+b-1, \dots, 2)$, we have $\Phi_\nu = \Phi$.

Consider $\mathcal{Q}_{a,b}$, and ν a permutation of $\{2, \dots, a+b\}$. Then, Φ_ν is a permutation on $J(\mathcal{Q}_{a,b})$ that partitions $J(\mathcal{Q}_{a,b})$ into orbits. Striker and Williams showed that the orbit structure⁴ of Φ_ν does not depend on the choice of ν .

Definition 10. Consider $\mathcal{Q}_{a,b}$, **promotion** is a permutation $\partial : J(\mathcal{Q}_{a,b}) \rightarrow J(\mathcal{Q}_{a,b})$, defined by: $\forall I \in J(\mathcal{Q}_{a,b}), \partial(I) = \sigma_{F_{a-1}} \circ \sigma_{F_{a-2}} \circ \dots \circ \sigma_{F_0} \circ \dots \circ \sigma_{F_{1-b}}(I)$.

As with rowmotion, Striker and Williams [5] define a generalized version of promotion.

Definition 11. [5] Consider $\mathcal{Q}_{a,b}$, and let ν be a permutation of $\{-b+1, \dots, a-1\}$. We define ∂_ν to be $\sigma_{F_{\nu(a+b-1)}} \circ \sigma_{F_{\nu(a+b-2)}} \circ \dots \circ \sigma_{F_{\nu(1)}}$. By Definition 10, for $\nu = (-b+1, \dots, a-1)$ we have $\partial_\nu = \partial$.

³Striker and Williams use the terminology “row” for what we call “rank” and “column” for what we call “file”.

⁴The orbit structure of a bijection f on a set S is the multiset of the sizes of the orbits that bijection f constructs on the set S .

As with rowmotion, for any permutation ν on files of any poset \mathcal{P} , ∂_ν will partition $J(\mathcal{P})$ to orbits. Again, Striker and Williams [5] showed that regardless of which ν we choose, $J(\mathcal{Q}_{a,b})$ will be partitioned into the same orbit structure by ∂_ν . Moreover, the orbit structures for ∂_ν and for Φ_ω are the same for any two permutations ν and ω :

Theorem 12. [5] *Consider the lattice $\mathcal{Q}_{a,b}$, for any permutation ν on $\{2, \dots, a+b\}$ and ω on $\{-b+1 \dots a-1\}$, there is an equivariant bijection between $J(\mathcal{Q}_{a,b})$ under Φ_ν and $J(\mathcal{Q}_{a,b})$ under ∂_ω .*

Permutations defined on combinatorial structures and the associated orbit structures became more interesting after Propp and Roby introduced a phenomenon called **homomesy** [7]. Propp and Roby also discussed some instances of homomesy by studying the actions of promotion and rowmotion on the set $J(\mathcal{Q}_{a,b})$. Homomesy has attracted many Combinatorics' attentions after it was defined and studied by Propp and Roby [8, 9, 10, 12, 13], and it is defined as follows:

Definition 13. [7] Consider a set S of combinatorial objects. Let $\tau : S \rightarrow S$ be a permutation that partitions S into orbits, and $f : S \rightarrow \mathbb{R}$ a statistic of the elements of S . We call the triple $\langle S, \tau, f \rangle$ **homomesic** (or we say it **exhibits homomesy**) if and only if there is a constant c such that for any τ -orbit $\mathcal{O} \subset S$ we have

$$\frac{1}{|\mathcal{O}|} \sum_{x \in \mathcal{O}} f(x) = c.$$

Equivalently, we can say f is homomesic or it exhibits homomesy in τ -orbits of S . If $c = 0$, the triple is called 0-mesic.

Proposition 14. *Consider a set S and permutation $\tau : S \rightarrow S$. If f_1, \dots, f_n are homomesic functions in τ -orbits of S , then any linear combination of the f_i s is also homomesic in τ -orbits of S .*

Theorem 15. [7] *Consider $f : J(\mathcal{Q}_{a,b}) \rightarrow \mathbb{R}$ defined as follows: for all $I \in \mathcal{Q}_{a,b}$, $f(I) = |I|$. Let $\partial, \Phi : J(\mathcal{Q}_{a,b}) \rightarrow J(\mathcal{Q}_{a,b})$ be the rowmotion and promotion operation. The triples $\langle J(\mathcal{Q}_{a,b}), \partial, f \rangle$ and $\langle J(\mathcal{Q}_{a,b}), \Phi, f \rangle$ exhibit homomesy.*

In this paper, we generalize Theorems 15 and 12 in the following sense:

Definition 16. Consider the poset \mathcal{P} to be one of $\mathcal{Q}_{a,b}$, \mathcal{U}_a or \mathcal{L}_a . For any permutation ν of $[a]$, we define the action **comotion**, $\mathcal{T}_\nu : J(\mathcal{P}) \rightarrow J(\mathcal{P})$ by: $\forall I \in J(\mathcal{P}), \mathcal{T}_\nu(I) = \sigma_{C_{\nu(a)}} \circ \sigma_{C_{\nu(a-1)}} \circ \dots \circ \sigma_{C_{\nu(1)}}(I)$.

The following proposition can be proved by applying Proposition 3 inductively.

Proposition 17. *Let \mathcal{P} be one of $\mathcal{Q}_{a,b}$, \mathcal{U}_a and \mathcal{L}_a , the action of promotion coincides with $\mathcal{T}_{(a,a-1,\dots,1)}$ and rowmotion coincides with $\mathcal{T}_{(1,2,\dots,a)}$.*

In what follows, Theorems 18, 19, 20 which are the main results of this paper will be stated. We will provide a roadmap to their proofs later in this introduction, and will complete the proof in Sections 1 and 2.

Theorem 18. *(Homomesy in $J(\mathcal{Q}_{a,b})$)*

1. For any arbitrary natural number a and ν a permutation on $[a]$, \mathcal{T}_ν partitions $J(\mathcal{Q}_{a,b})$ to orbits. The orbit structures of \mathcal{T}_ν on $J(\mathcal{Q}_{a,b})$ is independent of choice of ν .
2. Consider $I \in J(\mathcal{Q}_{a,b})$. We have the following homomesy results:
 - Let $g_{i,j}$, $1 \leq i \leq a$ and $1 \leq j \leq b$ be defined as follows:

$$g_{i,j} = \begin{cases} 1, & \text{if } |C_i(I)| = j \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

For an arbitrary permutation ν of $[a]$, $1 \leq i \leq a$ and $0 \leq j \leq b$, the function $d_{i,j} = g_{i,j} - g_{a+1-i,b-j}$ is 0-mesic in \mathcal{T}_ν -orbits of $J(\mathcal{Q}_{a,b})$.

- For all $1 \leq i \leq a$, let

$$s_{i,j} = \begin{cases} 1 & \text{if } |C_i(I)| + i = j \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

For any arbitrary permutation ν of $[a]$ and $1 \leq j \leq b$, $s_j = \sum_{i=1}^a s_{i,j}$ is homomesic in \mathcal{T}_ν -orbits of $J(\mathcal{Q}_{a,b})$. Moreover, the average of all s_j along an orbit is constant and equal to $\frac{a}{a+b}$.

In other words, for all j, l , $s_l - s_j$ is 0-mesic.

Any function $f : J(\mathcal{Q}) \rightarrow \mathbb{R}$ which is a linear combination of various s_i and d_i is homomesic in \mathcal{T}_ν -orbits of $J(\mathcal{Q}_{a,b})$.

Theorem 18 introduces a different family of permutations that produce the same orbit structure as Φ and ∂ ; hence, it generalizes Theorem 12. It also generalizes Theorem 15 because it introduces a class of permutations and statistics whose triple with $J(\mathcal{Q}_{a,b})$ exhibit homomesy. Moreover, it will provide another proof for Theorem 15. The main idea of our proof is the correspondence between comotion and

winching (See Definition 21). We will define winching and also its correspondence with comotion in Section 1. Then, we extend the definition of winching to winching with lower bounds and winching with zeros. Studying these two variations, helps us obtain homomesy results in $J(\mathcal{U}_a)$ and $J(\mathcal{L}_a)$.

Theorem 19. (*Homomesy in $J(\mathcal{U}_a)$*)

Let a be an arbitrary natural number and ν an arbitrary permutation of $[a]$. Consider $\mathcal{T}_\nu : J(\mathcal{U}_a) \rightarrow J(\mathcal{U}_a)$ as defined in Definition 16. For each $i \in [2a]$ let $[i, 2a] = i, i+1, \dots, 2a$ and $f : [2a] \rightarrow \mathbb{R}$ a function that has the same average in all $[i, 2a]$ where i is odd. Let $g : J(\mathcal{U}_a) \rightarrow \mathbb{R}$ be defined as: $\forall I \in J(\mathcal{U}_a), g(I) = \sum_{i=1}^a f(|C_i(I)| + 2i + 1)$. Then, the triple $\langle J(\mathcal{U}_a), \mathcal{T}_\nu, g \rangle$ exhibits homomesy.

Theorem 20. (*Homomesy in $J(\mathcal{L}_a)$*)

Let a be an arbitrary natural number and ν an arbitrary permutation of $[a]$ and $\mathcal{T}_\nu : J(\mathcal{L}_a) \rightarrow J(\mathcal{L}_a)$ be defined as in Definition 16. We have,

1. The orbit structures of \mathcal{T}_ν on $J(\mathcal{L}_{a,b})$ is independent from choice of ν .
2. For any $1 \leq i \leq a$ and $0 \leq j \leq a$ we define $s_{i,j} : J(\mathcal{L}_a) \rightarrow \mathbb{R}$ as follows:

$$s_{i,j} = \begin{cases} 1 & \text{if } |C_i| = j \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

For any $1 \leq j \leq a$ $s_j = \sum_{i=1}^a s_{i,j}$ is homomesic. Moreover, the average of all s_j along any \mathcal{T}_ν -orbit of $J(\mathcal{L}_{a,b})$ is the same. In other words, for all j, l $s_l - s_j$ is 0-mesic.

Moreover, any function $f : J(\mathcal{L}_a) \rightarrow \mathbb{R}$ which is a linear combination of various s_i is homomesic in \mathcal{T}_ν -orbits of $J(\mathcal{L}_a)$.

In Section 1 of this paper we introduce the permutation winching on the set of increasing sequences of length k . We show that there is a natural equivariant bijection between the set of ideals under comotion and the set of increasing sequences under winching.

Then, we introduce two different variations of winching and their correspondence with comotion in $J(\mathcal{U}_a)$ and $J(\mathcal{L}_a)$.

In Section 2 we will use the Theorems 18, 19 and 20 to show homomesy of some functions in the orbit structure produced by comotion in $J(\mathcal{Q}_{a,b})$, $J(\mathcal{U}_a)$ and $J(\mathcal{L}_a)$.

In Section 3 we will prove homomesy of a class of statistics when the permutation is winching and two different variations of it. These results have intrinsic interest because they are instances of homomesy.

Moreover, by the correspondence between winching and comotion proof of Theorems 18, 19, and 20 will be obtained.

1 Comotion, winching and their correspondence

In the previous section, we defined the action of comotion on the set of order ideals of a poset. In this section, we define winching and show a correspondence between winching on increasing sequences and comotion on $J(\mathcal{Q}_{a,b})$. Then, we define winching with lower bounds and winching with zeros. The former corresponds to comotion on $J(\mathcal{U}_a)$ and the later corresponds to comotion on $J(\mathcal{L}_a)$.

Definition 21. Let $S_{k,m}$ be the set of all k -tuples $x = (x_1, \dots, x_k)$ satisfying $0 < x_1 < x_2 < \dots < x_k < m+1$. We define the map $W_i : S_{k,m} \rightarrow S_{k,m}$, called winching on index i , by $W_i(x) = y = (y_1, y_2, \dots, y_k)$, where $y_j = x_j$ for $i \neq j$, and

$$y_i = \begin{cases} x_i + 1, & \text{if } x_i + 1 < x_{i+1}. \\ x_{i-1} + 1, & \text{otherwise.} \end{cases} \quad (5)$$

We assume that always $x_0 = 0$ and $x_{k+1} = m+1$.

Example 22. Let $\nu = (2, 3, 1, 4)$ and $x \in S_{4,7}$ be $x = (2, 3, 5, 7)$. Then, $W_\nu(x) = (1, 4, 6, 7)$.

Lemma 23. *There is a bijection $\alpha : J(\mathcal{Q}_{a,b}) \rightarrow S_{a,a+b}$ such that for any $I \in J(\mathcal{Q}_{a,b})$, $\alpha(\sigma_{C_j}(I)) = W_j(\alpha(I))$.*

Proof. Consider $I \in J(\mathcal{Q}_{a,b})$, we define $\alpha(I) = (\alpha_1, \dots, \alpha_a)$ as follows: for any $1 \leq j \leq a$, we have $\alpha_j(I) = |C_{a+1-j}(I)| + j$. Since $I \in J(\mathcal{Q}_{a,b})$, for any $j_1 < j_2$, take $|C_{j_1}(I)| \geq |C_{j_2}(I)|$. Therefore, $\alpha(I)$ is an increasing sequence.

Let C_j be $\{v_1, v_2, \dots, v_b\}$; $v_i = (j, i)$, and assume $|C_j(I)| = l$. We have, $n > l+1$, $\sigma_{v_n}(I) = I$, and for $n = l+1$, $\sigma_{v_n}(I) = I \cup \{v_n\}$ if and only if $|C_{j-1}| \geq l+1$. Also, $n < l$, $\sigma_{v_n}(I) = I - \{v_n\}$ if and only if $|C_{j+1}(I)| \leq n-1$. For boundary cases, we assume $|C_0| = b$ and $|C_b| = 0$. Letting $K = \sigma_{C_j}(I)$ we will have,

$$C_j(K) = \begin{cases} C_j(I) \cup \{v_{l+1}\}, & \text{if } |C_{j-1}(I)| \geq l+1. \\ C_j(I) - \{v_l, v_{l-1}, \dots, v_{p+1}\} (p = |C_{j+1}(I)|), & \text{otherwise.} \end{cases} \quad (6)$$

$$\Leftrightarrow |C_j(K)| + a+1-j = \begin{cases} l+1 + a+1-j, & \text{if } |C_{j-1}(I)| + a-j+2 \geq l+1 + a-j+2. \\ |C_{j+1}(I)| + a+1-j, & \text{otherwise.} \end{cases} \quad (7)$$

$$\Leftrightarrow \alpha_{a+1-j}(\sigma_{C_j}(I)) = \begin{cases} \alpha_{a-j+1}(I)+1, & \text{if } \alpha_{a-j+2}(I) > \alpha_{a-j+1}(I) + 1. \\ \alpha_{a-j}(I) + 1, & \text{otherwise.} \end{cases} \quad (8)$$

$$\Leftrightarrow \alpha_{a+1-j}(\sigma_{C_j}(I)) = W_{a+1-j}(\alpha(I)). \quad (9)$$

□

Corollary 24. *Consider an arbitrary natural number a , ν a permutation of $[a]$, and for any $x \in S_{a,a+b}$, let $W_\nu(x) = W_{\nu(a)} \circ W_{\nu(a-1)} \circ \cdots \circ W_{\nu(1)}(x)$. The bijection α introduced in Definition 23 satisfies the following property: $\alpha(\mathcal{T}_\nu(I)) = W_\nu(\alpha(I))$.*

Theorem 25. *Consider a natural number k and an arbitrary permutation ν of $[k]$. With $W_\nu : S_{k,m} \rightarrow S_{k,m}$ defined as above we will have,*

1. $W_\nu^m(x) = x$ for all $x \in S_{k,m}$.
2. *The orbit structure that winching produces on the set $S_{k,m}$ is the same as the orbit structure for rotation acting on the set of 2-colored necklaces with k white beads and $n - k$ black beads, and hence independent of choice of ν .⁵ (The orbit structure of necklaces is a classical problem in Combinatorics and the solution is a result of applying Pólya's Theorem [11].)*
3. *The following functions (and any linear combination of them) are homomesic in W_ν -orbits of $S_{k,m}$.*

- *Let $g_{i,j} : S_{k,m} \rightarrow \mathbb{R}$, $1 \leq i \leq k$ and $1 \leq j \leq m$ be defined as follows:*

$$g_{i,j}(x) = \begin{cases} 1, & \text{if } x_i = j \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

For any arbitrary $1 \leq i \leq k$ and $1 \leq j \leq m$, the function $d_{i,j} = g_{i,j} - g_{k+1-i,m+1-j}$ is 0-mesic in W_ν -orbits of $S_{k,m}$.

- *For an arbitrary $1 \leq j \leq m$, let $f_j : S_{k,m} \rightarrow \mathbb{R}$ be defined by:*

$$f_j(x) = \begin{cases} 1, & \text{if } j \in x \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

⁵The definition of rotation acting on the set of 2-colored necklaces is presented in Section 3 (Definition 47).

For any $1 \leq j \leq m$, the triple $\langle S_{k,m}, W_\nu, f_j \rangle$ is homomesic and the average of f_j along W_ν orbits is k/m .

We will prove the above theorem in the next section. Given the bijection in Corollary 24, Theorem 18 is a straightforward conclusion of Theorem 25. In addition, Theorem 15 can be concluded from the above theorem. In fact, a more general statement is shown in the next section (Corollary 34).

The following variation of winching is called **winching with lower bounds** and it corresponds to comotion on $J(\mathcal{U}_a)$.

Definition 26. Consider the sequence of lower bounds $l = (l_1, \dots, l_k)$, $0 < l_1 < \dots < l_k < m+1$ and $S'_{k,m} = \{(x_1, x_2, \dots, x_k) \in S_{k,m} | x_i \geq l_i\}$, where $S_{k,m}$ is the set defined in Definition 21. For any index $i \in [k]$, we define the map $\underline{W}_i : S'_{k,m} \rightarrow S'_{k,m}$ called winching with lower bounds l on index i by

$$\forall w \in S'_{k,m} \quad \underline{W}_i(w) = \max\{W_i(w), l_i\},$$

where W_i is the action of winching on index i (Definition 21). Having \mathcal{U}_a be the poset which is defined in Definition 5, we will have:

Lemma 27. *There is a bijection $\beta : J(\mathcal{U}_a) \rightarrow S'_{a,2a}$ such that for the lower bounds $l = (1, 3, 5, \dots, 2a-1)$, we have: for any $I \in J(\mathcal{U}_a)$, $\beta(\sigma_{C_j}(I)) = \underline{W}_j(\beta(I))$.*

Proof. Fix arbitrary a and consider $I \in J(\mathcal{U}_a)$, we define $\beta(I) = (\beta_1, \dots, \beta_a)$ as follows: for any $1 \leq j \leq a$, $\beta_j(I) = |C_{a+1-j}(I)| + 2j - 1$. Considering the ideal $I' \in J(\mathcal{Q}_a)$, $I' = I \cup (\mathcal{Q}_{a,a} - \mathcal{U}_a)$, we will have, $\beta(I) = \alpha(I')$. Hence, β is an increasing sequence. Since $\sigma_{C_j}(I) = \sigma_{C_j}(I') - (\mathcal{Q}_{a,a} - \mathcal{U}_a)$ we have,

$$\beta_j(\sigma_{C_{a+j-1}}(I)) = |\sigma_{C_{a+j-1}}(I)| + 2j - 1 = |\sigma_{C_{a+j-1}}(I') - (\mathcal{Q}_{a,a} - \mathcal{U}_a)| + 2j - 1 \quad (12)$$

$$\Rightarrow \beta_j(\sigma_{C_{a+j-1}}(I)) = \max\{|\sigma_{C_{a+j-1}}(I')| - j + 1, 0\} + 2j - 1 = \max\{|\sigma_{C_{a+j-1}}(I')| + j, 2j - 1\} \quad (13)$$

$$\Rightarrow \beta_j(\sigma_{C_{a+j-1}}(I)) = \max\{(W_j(\alpha(I')))_j, 2j - 1\} = \max\{(W_j(\beta(I)))_j, 2j - 1\}. \quad (14)$$

□

Corollary 28. Consider any arbitrary permutation $\nu : [a] \rightarrow [a]$, the action $\mathcal{T}_\nu : J(\mathcal{U}_a) \rightarrow J(\mathcal{U}_a)$ and $I \in J(\mathcal{U}_a)$. For any $x \in S_{a,2a}$, let the lower bounds be $l = (1, 3, \dots, 2a-1)$. Then: $\underline{W}_\nu(x) = \underline{W}_{\nu(a)} \circ \underline{W}_{\nu(a-1)} \circ \dots \circ \underline{W}_{\nu(1)}(x)$. Bijection β introduced in 27 satisfies the following property: $\beta(\mathcal{T}_\nu(I)) = \underline{W}_\nu(\beta(I))$.

Theorem 29. Let ν be an arbitrary permutation of $[a]$. Consider $\underline{W}_\nu : S'_{a,b} \rightarrow S'_{a,b}$ with lower bounds (l_1, l_2, \dots, l_a) . For each $i \in [a+b]$ let $[i, a+b] = i, i+1 \dots a+b$ and $f : [a+b] \rightarrow \mathbb{R}$ a function that has the same average in all $[l_i, a+b]$, $1 \leq i \leq a$. Let $g : S'_{a,b} \rightarrow \mathbb{R}$ be defined as, $g(x) = \sum_{i=1}^a f(x_i)$. Then, the triple $\langle S'_{a,b}, \underline{W}_\nu, g \rangle$ exhibits homomesy.

We now define the action of **winching with zeros** to study homomesy in $J(\mathcal{L}_a)$.

Definition 30. Let S_n be the set of all n -tuples $x = (x_1, \dots, x_n)$ such that for some $0 \leq k \leq n$ $x_1 = x_2 = \dots = x_k = 0$ and $1 \leq x_{k+1} < x_{k+2} < \dots < x_n \leq n$. We define the map $\text{WZ}_i : S_n \rightarrow S_n$, called winching with zeros on index i to be

$$\text{WZ}_i(x) = \begin{cases} x_i+1 & \text{if } x_i+1 < \min\{x_{i+1}, n+1\}; \\ x_{i-1}+1 & \text{if } 1 < i \text{ and } 0 < x_{i-1}; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 31. There is a bijection $\gamma : J(\mathcal{L}_a) \rightarrow S_a$ such that: for any $I \in J(\mathcal{L}_a)$, $\gamma(\sigma_{C_j}(I)) = \text{WZ}_j(\gamma(I))$.

Proof. Fix an arbitrary natural number a and consider $I \in J(\mathcal{L}_a)$. We define, $\gamma(I) = (\gamma_1, \gamma_2, \dots, \gamma_a)$ as follows: for $1 \leq j \leq a$, $\gamma_j(I) = |C_{a+1-j}(I)|$. For any $j_1 < j_2$, we have $|C_{j_1}(I)| > |C_{j_2}(I)|$. Hence, γ will be an increasing sequence.

Let $C_j = \{v_j, v_{j+1}, \dots, v_a\}$ where for $j \leq i \leq a$, $v_i = (j, i)$. Assume $|C_j(I)| = l$, which means $C_j(I) = \{v_j, v_{j+1}, \dots, v_{j+l-1}\}$. For $n > j+l$, $\sigma_{v_n}(I) = I$. We have three cases: if $n = j+l$, we will have $\sigma_{v_n}(I) = I \cup \{v_n\}$ if and only if $(j-1, j+l) \in I$ i.e. $|C_{j-1}(I)| > l+1$. If $C_{j+1}(I) = 0$, $\sigma_{C_j}(I) = I - C_j(I)$. And if $\sigma_{C_j}(I) > 0$, then $\sigma_{C_j}(I) = I - \{v_{k+1}, \dots, v_{j+l-1}\}$, where $k = |C_{j+1}(I)|$. Letting $\sigma_{C_j}(I) = K$, we will have:

$$C_j(K) = \begin{cases} C_j(I) \cup \{v_{j+l}\}, & \text{if } |C_{j-1}(I)| > l+1. \\ \emptyset & \text{if } |C_{j-1}(I)| \leq l+1 \text{ and } |C_{j+1}(I)| = 0. \\ C_j(I) - \{v_{k+1}, v_{k+2}, \dots, v_{j+l-1}\}, & \text{otherwise.} \\ k = |C_{j+1}(I)| > 0 \end{cases} \quad (15)$$

$$|C_j(K)| = \begin{cases} l+1, & \text{if } |C_{j-1}(I)| > l+1. \\ 0 & \text{if } |C_{j-1}(I)| \leq l+1 \text{ and } |C_{j+1}(I)| = 0. \\ k+1, & \text{otherwise.} \\ k = |C_{j+1}(I)| > 0 \end{cases} \quad (16)$$

$$\gamma_j(K) = \begin{cases} \gamma_j(I) + 1, & \text{if } \gamma_{j+1}(I) > l+1. \\ 0 & \text{if } \gamma_{j+1}(I) \leq l+1 \text{ and } \gamma_{j-1}(I) = 0. \\ \gamma_{j-1} + 1, & \text{otherwise.} \end{cases} \quad (17)$$

$$\Leftrightarrow \gamma_{a+1-j}(\sigma_{C_j}(I)) = WZ_{a+1-j}(\gamma(I)). \quad (18)$$

□

Corollary 32. Consider any arbitrary natural number $[n]$ and permutation ν on n , the action $\mathcal{T}_\nu : J(\mathcal{L}_a) \rightarrow J(\mathcal{L}_a)$, and $I \in J(\mathcal{L}_a)$. For any $x \in S_a$, we will have: $WZ_\nu(x) = WZ_{\nu(a)} \circ WZ_{\nu(a-1)} \circ \dots \circ WZ_{\nu(1)}(x)$. The bijection γ introduced in 31 satisfies the following property: $\gamma(\mathcal{T}_\nu(I)) = WZ_\nu(\gamma(I))$.

Theorem 33. Consider an arbitrary natural number n and an arbitrary permutation ν of $[n]$. With $WZ_\nu : S_n \rightarrow S_n$ defined as above we will have,

1. $WZ_\nu^{2n}(x) = x$ for all $x \in S_n$.
2. For an arbitrary $1 \leq j \leq n$, let $f_j : S_n \rightarrow \mathbb{R}$ be defined by:

$$f_j(x) = \begin{cases} 1, & \text{if } j \in x \\ 0, & \text{otherwise.} \end{cases} \quad (19)$$

The triple $\langle S_n, WZ_\nu, f_j \rangle$ is homomesic and the average of f_j along WZ_ν -orbits is $1/2$. Moreover, any linear combination of f_j s is homomesic in WZ_ν -orbits of S_n .

We will prove the above theorem in Section 3. Given the bijection in Corollary 24, Theorem 20 is a straightforward consequence of Theorem 33.

2 Some homomesy results in the comotion-orbits of $J(\mathcal{Q}_{a,b})$, $J(\mathcal{L}_a)$, and $J(\mathcal{U}_a)$.

The following homomesy results can be easily verified using Theorem 18.

Corollary 34. Let \mathcal{P} be $\mathcal{Q}_{a,b}$ or \mathcal{L}_a . Consider an arbitrary natural number a , an arbitrary permutation ν , and $\mathcal{T}_\nu : J(\mathcal{P}) \rightarrow J(\mathcal{P})$ as defined in 16. We define the size function, $f : J(\mathcal{P}) \rightarrow \mathbb{R}$ as, $\forall I, f(I) = |I|$. The triple $\langle J(\mathcal{P}), \mathcal{T}_\nu, f \rangle$ is homomesic for any choice of ν .

Proof. For $\mathcal{P} = \mathcal{Q}_{a,b}$, $f = \sum_{i=1}^a i s_i - a(a+1)/2$. For $\mathcal{P} = \mathcal{L}_a$, $f = \sum_{i=1}^a i s_i$. In both cases f is a linear combination of f_i using Theorems 18 and 20 we will have the result. \square

Corollary 35. Consider the lattice $\mathcal{Q}_{a,b}$ and an arbitrary permutation ν of $[a]$. Let $x \in [a] \times [b]$. We define the antipodal function $A : [a] \times [b] \rightarrow [a] \times [b]$ by $A(x) = y$ where $x = (i, j) \Leftrightarrow y = (a-i+1, b-j+1)$. For $I \in J(\mathcal{Q}_{a,b})$ and $x \in [a] \times [b]$, we define the characteristic function $\mathcal{I}_I(x) : [a] \times [b] \rightarrow \{0, 1\}$ as follows:

$$\mathcal{I}_I(x) = \begin{cases} 1 & \text{if } x \in \mathcal{I} \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

For any arbitrary $x \in [a] \times [b]$ let $h : J(\mathcal{Q}_{a,b}) \rightarrow \{0, 1, -1\}$ be given by $h(I) = \mathcal{I}_I(x) - (1 - \mathcal{I}_I(A(x)))$. Then h is 0-mesic in \mathcal{T}_ν -orbits of $J(\mathcal{Q}_{a,b})$. In other words, we have **central antisymmetry**, i.e. the average of number of ideals that contain x is equal to the number of ideals that do not contain $A(x)$.

Proof. Consider arbitrary $I \in \mathcal{Q}_{a,b}$ and $x = (x_1, x_2) \in [a] \times [b]$. Then

$$\begin{aligned} \mathcal{I}_I(x) = 1 &\Leftrightarrow (x_1, x_2) \in \mathcal{I} \Leftrightarrow |C_{x_1}(I)| \geq x_2 \\ \Rightarrow \mathcal{I}_I(x) &= \sum_{j=x_2}^b g_{x_1, j}. \end{aligned} \quad (21)$$

Similarly,

$$\begin{aligned} 1 - \mathcal{I}_I(A(x)) = 1 &\Leftrightarrow (a - x_1 + 1, b - x_2 + 1) \notin \mathcal{I} \Leftrightarrow |C_{a-x_1+1}(I)| < b - x_2 + 1 \Leftrightarrow \\ |C_{a-x_1+1}(I)| &\leq b - x_2 \Rightarrow 1 - \mathcal{I}_I(A(x)) = \sum_{j=0}^{b-x_2} g_{a-x_1+1, j} = \sum_{j=x_2}^b g_{a-x_1+1, b-j}. \end{aligned} \quad (22)$$

By Equations 21 and 22, we have $h_x(\mathcal{I}) = \sum_{j=x_2}^b g_{x_1, j} - g_{a-x_1+1, b-j}$. Employing Theorem 18 we deduce that h_x is 0-mesic for any arbitrary $x \in [a] \times [b]$. \square

Corollary 36. Let \mathcal{P} be one of \mathcal{Q}_a or \mathcal{U}_a . Consider arbitrary $I \in J(\mathcal{P})$. We denote the **rank-alternating** cardinality of I by $\mathcal{R}(I)$ and we define it as $\mathcal{R}(I) = \sum_{(i,j) \in I} (-1)^{i+j}$. The triple $\langle J(\mathcal{P}), \mathcal{T}_\nu, \mathcal{R} \rangle$ is homomesic for any arbitrary permutation ν of $[a]$.

Proof. We will first consider the case when $I \in J(\mathcal{Q}_{a,b})$. In this case we have:

$$\begin{aligned}
2 \mathcal{R}(I) &= \sum_{x=(i,j) \in \mathcal{P}} (-1)^{i+j} \mathcal{I}_I(x) = \sum_{x=(i,j)} (-1)^{i+j} \mathcal{I}_I(x) + \sum_{x=(i,j)} (-1)^{i+j} \mathcal{I}_I(x) \\
&\Rightarrow 2 \mathcal{R}(I) = \sum_{x=(i,j) \in \mathcal{X}} (-1)^{i+j} \mathcal{I}_I(x) + (-1)^{2a-(i+j)+2} \mathcal{I}_I(A(x)) \\
&= \sum_{x=(i,j) \in \mathcal{X}} (-1)^{i+j} h(x) + 1.
\end{aligned} \tag{23}$$

In the case where $I \in J(\mathcal{U}_a)$ we have:

$$\mathcal{R}(I) = (-1)^{a+1} \sum_{i: |C_i| \text{ odd}} 1.$$

We define the function $f : \mathbb{N} \rightarrow \{0, 1\}$ as follows: $f(x) = 1$ iff x odd, $f(x) = 0$ otherwise. Note that the average of f in any $[i, 2a]$ that i is odd is equal to $1/2$. Therefore, by Theorem 19 we will have the result. □

3 Homomesy in winching

In this section we will prove Theorems 25, 29, and 33. The concepts of **tuple board** and **snake** are the prime definitions of this section, and they help us understand the orbit structure and homomesy in winching.

Fix k , for arbitrary ν a permutation of $[k]$, let \mathcal{F}_ν be one of W_ν , \underline{W}_ν or WZ_ν . Let $S = S_k$ if $\mathcal{F} = WZ_\nu$ and $S = S_{k,m}$ otherwise. We define a tuple board as follows:

Definition 37. Consider $x \in S$ and ν a permutation of $[k]$. We write $x, \mathcal{F}_\nu(x), \mathcal{F}_\nu^2(x), \dots$ in separate, consecutive rows as depicted below. Let $TB(x) = [x^1, x^2, \dots]$ be such a table, where $TB(i, \cdot) = x^i = (x_1^i, \dots, x_k^i)$ and $x^i = \mathcal{F}_\nu^{i-1}(x)$. We will have a board looking as follows:

row 1 (x^1)	x_1^1	\dots	x_k^1
row 2 (x^2)	x_1^2	\dots	x_k^2
row 3 (x^3)	x_1^3	\dots	x_k^3
\dots	\dots	\dots	\dots
\dots	\dots	\dots	\dots

Figure 2. A tuple board.

$TB(x)$ is called the tuple board of x . Since \mathcal{F}_ν is a permutation, there is some n such that $\mathcal{F}_\nu^{n+1}(x) = x$. Therefore, we can also define a cylinder corresponding to the orbit containing x :

Consider \mathcal{O} , an \mathcal{F}_ν -orbit of S which is produced by applying \mathcal{F}_ν consecutively to x . We define the **tuple cylinder** $TS(\mathcal{O})$ to be the cylinder that is produced by attaching the first and the $n+1$ st row of $TB(x)$. Since \mathcal{O} is an orbit it is more natural to think of a tuple board as a cylinder. We will use the terms interchangeably in this text.

Notice that any cell in a tuple board contains a number from the set $\{0, 1, 2, \dots, m\}$. In what comes in the following we will introduce the notion of snakes. Given a tuple board T , any snake in it, is a sequence of adjacent cells in T that contain the numbers $1, 2, \dots, m$. The mathematical definition of a snake comes in the following:

Definition 38. For arbitrary $\nu = (\nu_1, \nu_2, \dots, \nu_k)$ a permutation of $[k]$ and $x \in S$, let $TB = TB(x)$ be the tuple board of x as defined in Definition 37. Considering $T = \{TB(i, j) | 1 \leq i \leq n, 1 \leq j \leq k\}$, we define a **snake** $s = (s_f, s_{f+1}, \dots, s_t)$ as follows: s is a maximal sequence of s_i s such that each s_i is a cell in the tuple board containing i , and for $i > f$, $s_i = \mathcal{M}(s_{i-1})$, where \mathcal{M} is defined as follows:

$$\mathcal{M}(T(i, j)) = \begin{cases} T(i+1, j) & \text{if } T(i+1, j) = T(i, j) + 1. \\ T(i, j+1) & \text{if } T(i, j+1) = T(i, j) + 1, \quad T(i+1, j) \neq T(i, j) + 1 \\ & \text{and } \nu(j) < \nu(j+1). \\ T(i+1, j+1) & \text{if } T(i+1, j+1) = T(i, j) + 1, T(i+1, j) \neq T(i, j) + 1 \\ & \text{and } \nu(j) > \nu(j+1). \end{cases} \quad (24)$$

Definition 39. Consider $T = TB = [x^1, x^2, \dots, x^n]$ as defined previously for $x \in S$. In what follows row numbers in a tuple board are understood modulo n .

Consider s a snake in T . We define **snake map** \mathcal{S} , a function that associates any snake with an element in \mathbb{N}^k as follows: for an arbitrary snake s , $\mathcal{S}(s) = (c_1, c_2, \dots, c_k)$, where $c_j = |\{i | T(i, j) \in s\}|$.

...
1	2	?	?	?
?	3	?	?	?
?	4	5	?	?
?	?	6	?	?
?	?	7	?	?
?	?	?	8	9
?	?	?	?	10
...

...
?	4	?	?	?
?	?	5	?	?
?	?	6	?	?
?	?	7	?	?
?	?	8	9	10
?	?	?	?	?
...
...

...
...
0	1	?	?	?
?	?	2	?	?
?	?	3	4	5
?	?	?	?	?
?	?	?	?	?
...
...

Figure. 3.

A tuple board corresponding to W_ν
 $x \in S_{5,10}$ and
 $\nu = (1, 2, 4, 3, 5)$.
The snake map is $(1, 3, 3, 1, 2)$.

A tuple board corresponding to W_ν
with lower bounds $(2, 4, 6, 7, 8)$.
 $x \in S_{5,10}$ and $\nu = (1, 3, 2, 4, 5)$.
The snake map is $(0, 1, 4, 1, 1)$.

A tuple board corresponding
to WZ_ν . $x \in S_5$ and
 $\nu = (1, 3, 2, 4, 5)$
The snake map is $(0, 1, 2, 1, 1)$.

3.1 Proof of Theorem 25

In this subsection we prove Theorem 25.

Definition 40. Let $\bar{W}_i : S_{k,m} \rightarrow S_{k,m}$ be the following map: $\forall x = (x_1 \dots x_k) \in S_{k,m}$, $\bar{W}_i(x) = y = (y_1, y_2, \dots, y_k)$ where $\forall j \neq i, y_j = x_j$, and

$$y_i = \begin{cases} x_{i+1} - 1, & \text{if } x_i = x_{i-1} + 1. \\ x_i - 1, & \text{otherwise.} \end{cases} \quad (25)$$

Note that $\forall x \in S_{k,m}$, $\bar{W}_i \circ W_i(x) = x$. We call \bar{W}_i **inverse winching** at index i .

Definition 41. For ν an arbitrary permutation of $[k]$, $\bar{W}_\nu : S^{k,m} \rightarrow S^{k,m}$ is defined by $\bar{W}_\nu = \bar{W}_{\nu(1)} \circ \bar{W}_{\nu(2)} \circ \dots \circ W_{\nu(k)}^r$ and we have $\forall x \in S_{k,m}$, $\bar{W}_\nu(W_\nu(x)) = x$.

Lemma 42. Any snake in a tuple cylinder $TS(\mathcal{O})$ (where \mathcal{O} is a W_ν -orbit of $S_{k,m}$) is of length m , starts in the first column of the cylinder with s_1 , and ends in the last column of the tuple cylinder with s_m .

Proof. Consider some $x \in \mathcal{O}$ and a snake s in the tuple board $T = TB(x) = [x^1, \dots, x^n]$. We assume that $s = (s_f, \dots, s_t)$. Having, $x^{i+1} = W_\nu(x^i)$, it is easy to verify that unless $t = m$, we can find a cell in T to expand s . Similarly, since $x^{i-1} = \bar{W}(x^i)$. If $\nu(j) < \nu(j+1)$, we can see: unless $f = 1$, the snake s can be expanded.

□

Definition 43. Let $H : [m]^{[k]} \rightarrow [m]^{[k]}$ be defined as follows: $\forall x = (x_1, \dots, x_k)$, $H(x) = y = (y_1, \dots, y_k)$ where $\forall 1 \leq i < k, y_i = x_{i+1}$ and $y_k = x_1$. We call H the **left shift operator**.

Lemma 44. Let $(p, 1)$ and $(q, 1)$ ($p < q$) be two cells of tuple board T with value 1, such that there is no $p < i < q$ with $T(i, 1) = 1$. Consider the snake $s^p = (s_1^p \dots s_m^p)$ starting with $s_1^p = T(p, 1)$ and $\mathcal{S}(s^p) = c^p = (c_1^p, \dots, c_k^p)$ its snake map; and similarly consider the snake s^q and its snake map $\mathcal{S}(s^q) = c^q$ starting at $T(q, 1)$. Then,

- If $T(i, j) \in s^p$, we have the following:

- $T(i+1, j) \notin s^p \Rightarrow T(i+1, j) \in s^q$.
- If $j > 1$ then, $T(i, j-1) \notin s^p \Rightarrow T(i, j-1) \in s^q$.

In other words there is no gap between two consecutive snakes in the tuple board.

- We have $c^q = H(c^p)$.

Proof. In order to prove this lemma we fix $\nu = (1, 2, \dots, k)$. The proof will be similar for any arbitrary permutation ν . To make notation simpler we drop the subscript from W meaning $\nu = (1, 2, \dots, k)$.

Suppose that we have the action of winching $W_{(1,2,\dots,k)}$ on $x \in S_{k,m}$ making the orbit \mathcal{O} in $S_{k,m}$. Moreover, suppose the tuple board corresponding to x (equivalently, the tuple cylinder corresponding to \mathcal{O}) is $T = TB(x) = [x^1, x^2, \dots, x^n]$ where $x^i = W^{i-1}(x)$ is as defined in Definition 37.

Claim 1. $c_1^p = q - p$.

Since $T(q, 1) \notin s^p$, $c_1^p \leq q - p$. Moreover $c_1^p = c_1 \leq q - p$ implies $s_{c_1+1}^p = T(p+c_1-1, 2) = c_1 + 1$ meaning $x_2^{c_1} = c_1 + 1$ and $x_1^{c_1} = c_1$. We have $x^{c_1+1} = W(x_1^c)$, and hence $x_1^{c_1+1} = 1$, and $T(c_1 + p, 1) \in s^q \Rightarrow c_1 + p = q \Rightarrow c_1 = q - p$.

Note that Claim 1 implies that there is no gap between the two snakes in column 1. (See Figure 4).

Claim 2. $c_1^q = c_2^p$. For simplicity, we denote c_1^p by c_1 and c_2^p by c_2 .

We have $s_1^p = T(p, 1) = 1$, $s_2^p = T(p+1, 1), \dots, s_{c_1}^p = T(p+c_1-1, 1)$. (See Figure 4)

Then, for all $1 \leq i \leq c_2$:

$$\begin{aligned} s_{c_1+i}^p &= T(p+c_1-1+(i-1), 2) \\ \Rightarrow s_{c_1+i}^p &= T(q+(i-2), 2) & (\text{Since } q = p + c_1) \\ \Rightarrow x_2^{c_1+i-1} &= c_1 + i \end{aligned} \quad (26)$$

We also have

$$s_{c_1+c_2+1}^p = T(q+c_2-2, 3) = T(p+c_1+c_2-2, 3) \Rightarrow x_3^{c_1+c_2-1} = c_1 + c_2 + 1. \quad (27)$$

Now consider s^q . For all $i, 1 \leq i \leq c_2 - 1$ we have that if $x_1^{c_1+i} = i$,

$$\left. \begin{aligned} x_1^{c_1+i} &= i \\ x_2^{c_1+i} &= c_1 + i + 1 > i \\ W(x^{c_1+i}) &= x^{c_1+i+1} \end{aligned} \right\} \Rightarrow x_2^{c_1+i+1} = i + 1 \quad (28)$$

Therefore,

$$x_1^{c_1+1} = 1 \Rightarrow \forall i, 1 \leq i \leq c_2 - 1, \mathcal{M}(s_i^q) = T(q+i, 1) \Rightarrow \forall 1 \leq i \leq c_2, s_i^q = T(q+i-1, 1). \quad (29)$$

From Equation 28 we can conclude $x_1^{c_1+c_2} = c_2$. By Equations 27 and 26, and the fact that $W(x^{c_1+c_2-1}) = x^{c_1+c_2}$, we have $x_2^{c_1+c_2} = c_2 + 1$. Hence, $\mathcal{M}(s_{c_2}^q) = T(q + c_2 - 1, 2)$.

It follows that $c_1^q = c_2^p$. Moreover, $T(i, 2) \in s^p \Rightarrow T(i, 1) \in s^q$ and $T(i - 1, 2) \in s^q$ for any i (if they are not already in s^p).

Very similar to the proof of Claim 2, the following can be proved using the definitions:

Claim 3. Let $r < k$ with $\forall l, 1 \leq l < r - 1$, $c_l^q = c_{l+1}^p$, then $c_r^q = c_{r+1}^p$.

Having Claims 2 and 3, by employing induction we can show: for all $i, 1 \leq i < k - 1$, $c_i^q = c_{i+1}^p$, and that there is no gap between the snakes in any of the columns. Furthermore, since all snakes have the same length, $c_k^q = c_1^p$. \square

Proof of Theorem 25, Part 1. Consider an $n \times k$ tuple board T such that $T = TB(x)$ and $x \in S_{k,m}$. Let's assume that $n \geq m$ (if $n < m$, append enough copies of T to it until $n \geq m$). Let s^1 be the snake that covers $T(1, 1)$, s^2 the next snake immediately below s^1 , and s^i the last snake right below s^{i-1} . Letting $\mathcal{S}(s^1) = c = (c_1, c_2, \dots, c_k)$, we have $\mathcal{S}(s^i) = H^{i-1}(c)$. The numbers in the first column of T will be: $x_1, x_1 + 1, \dots, x_1 + c_1 - 1, 1, 2, \dots, c_2, 1, 2, \dots, c_3, \dots$. Since $\sum_{i=1}^k c_i = m$, the $m + 1$ st number in the first column will be x_1 . Similarly, for each column i , the $m + 1$ st element will be x_i . Thus, $W^{m+1}(x) = x$. \square

Corollary 45. *The above reasoning also shows there are exactly k snakes covering an $m \times k$ tuple cylinder.*

Corollary 46. *Fix k and n and ν a permutation of $[k]$. To each tuple cylinder T of size $k \times n$ corresponding to a W_ν -orbit, we can assign a sequence $c = (c_1, c_2, \dots, c_k)$, satisfying $\sum_{i=1}^k c_i = n$ where T is covered by snakes s_1, s_2, \dots, s_k and for all $1 \leq j \leq k$, there is an i such that $\mathcal{S}(s_j) = H^i(c)$. Since filling the first column of the cylinder will impose the other numbers, this correspondence is a one to one mapping.*

		...				
row p (x^1)	1	...				
row $p+1$ (x^2)	2	...				
row $p+c_1-1$ (x^{c_1})	c_1	c_1+1	...			
row q (x^{c_1+1})	1	c_1+2	...			
row $q+c_2-1$ ($x^{c_1+c_2-1}$)	c_2	c_2+1				
row $q+c_2$ ($x^{c_1+c_2}$)	1	c_2+2	...			

Figure. 4. Snakes in a tuple board of $W_{(1,2,\dots,a)}$.

In order to prove Part 2 of Theorem 25, we present the definition of rotation on 2-colored necklaces with k white beads and $n - k$ black beads, then we proceed to the proof:

Definition 47. Let $N_{k,m}$ the set of all k -tuples (x_1, x_2, \dots, x_k) satisfying $1 \leq x_1 < x_2 < \dots < x_k \leq m$ and $\sum_{i=1}^k x_i = m$. The action of rotation on this set is defined as $R : N_{k,m} \rightarrow N_{k,m}, \forall x \in N_{k,m}, R(x) = y$, where $y = (x_1+1, x_2+1, \dots, x_k+1)$ if for all $i, x_i < m$. Or $y = (1, x_1+1, \dots, x_{k-1}+1)$ if $x_k = m$.

Lemma 48. There is a map $\mathcal{K} : S_{k,m} \rightarrow N_{k,m}$ satisfying $\forall x \in S_{k,m}, R(\mathcal{K}(x)) = \mathcal{K}(\bar{W}_\nu(x))$.

Proof. Consider arbitrary $x \in S_{k,m}$ and $T = TB(x)$ as in Definition 37. Let s be the snake covering $T(1, 1)$. For $\mathcal{S}(s) = (c_1, c_2, \dots, c_k)$, we define $\mathcal{K}(x) = (y_1, y_2, \dots, y_k) \in N_{k,m}$, where $y_1 = c_1 - x_1 + 1$, and for $2 \leq i \leq k, y_i = y_{i-1} + c_i$.

Note that $R(\mathcal{K}(x)) = \mathcal{K}(\bar{W}(x))$ if and only if $R(\mathcal{K}(W(x))) = \mathcal{K}(x)$. Let $T = TB(x), T' = TB(W(x))$. Let s be the snake in T covering $T(1, 1)$, $c = \mathcal{S}(s)$, and similarly let s' be the snake in T' covering $T'(1, 1)$, $c' = \mathcal{S}(s')$, and $W(x) = z$. Either $c = c'$ and $x_1 + 1 = z_1$ or $c' = H(c)$, $x_1 = c_1$, and $z_1 = 1$.

$$R(\mathcal{K}(z)) = R(y_1, y_2, \dots, y_k); y_1 = c'_1 - z_1 + 1, y_{i+1} = y_i + c'_{i+1} \quad (30)$$

$$= \begin{cases} R(y_1, y_2, \dots, y_k); y_1 = c_1 - (x_1 + 1) + 1, y_{i+1} = y_i + c_{i+1} \\ R(y_1, y_2, \dots, y_k); y_i | y_1 = c_2 - 1 + 1, y_{i+1} = y_i + c_{i+2} \end{cases} \quad (31)$$

$$= \begin{cases} R(y_1, y_2, \dots, y_k); y_1 = c_1 - x_1, y_{i+1} = y_i + c_{i+1} \\ R(y_1, y_2, \dots, y_k); y_1 = c_2, y_{i+1} = y_i + c_{i+2 \pmod k} \end{cases} \quad (32)$$

$$= \begin{cases} \{(y'_1, y'_2, \dots, y'_k); y_1 = c_1 - x_1 + 1, y_{i+1} = y_i + c_{i+1}\} & \text{because } \forall i, y_i < m \\ R(\{y_i | y_i = c_{i+1} (1 \leq i \leq k-1), y_k = m\}) = (1, 1+c_2, 1+c_2+c_3 \dots 1 + \sum_{i=1}^k c_k) \end{cases} \quad (33)$$

$$= \mathcal{K}(x) \quad (34)$$

□

Example 49. Consider $x = (2, 3, 4, 6) \in S_{4,7}$, $W(x) = (1, 2, 5, 6)$. In Figure 5, $TB(x)$ and $TB(W(x))$ are depicted. We see that $\mathcal{K}(W(x)) = (1, 4, 5, 7)$, and $\mathcal{K}(x) = (1, 2, 5, 6)$. Note that $R(1, 4, 5, 7) = (1, 2, 5, 6)$.

$T = TB(2, 3, 4, 6) :$	2	3	4	6
	1	2	5	7
	1	3	6	7
	2	4	5	6
	3	4	5	7
	1	2	6	7
	1	3	4	5

$T' = TB(1, 2, 5, 7) :$	1	2	5	7
	1	3	6	7
	2	4	5	6
	3	4	5	7
	1	2	6	7
	1	3	4	5
	2	3	4	6

Figure. 5. The snake covering $T(1, 1)$ has snake map $(2, 1, 3, 1)$. Hence, $\mathcal{K}(x) = (1, 2, 5, 6)$. The snake covering $T'(1, 1)$ has snake map $(1, 3, 1, 2)$. Hence, $\mathcal{K}(W(x)) = (1, 4, 5, 7)$.

Proof of Theorem 25, Part 2. Having Lemma 48, we conclude that the orbit structures of $\langle N_{k,m}, R \rangle$ and $\langle S_{k,m}, W \rangle$ are the same.

Lemma 50. Let T be an $m \times k$ tuple board. Consider the column r : $T_r = \{T(i, r)\}$. For any $1 \leq r \leq k$, there exists a one-to-one function $\mathcal{F} : T_r \rightarrow T_{k+1-r}$, satisfying $\mathcal{F}(x) = m+1-x$.

Proof. For any r , we construct a mapping from $\{\cup_{t=1}^r T_t\}$ to $\{\cup_{t=k-r+1}^k T_t\}$. Consider a number x in T_r . Let it be the l th element in T_r , covered by a snake having snake map $p = (c_1, c_2, \dots, c_r, \dots, c_k)$. Consider the snake with snake map $p' = (c_{r+1}, \dots, c_1, c_2, \dots, c_r)$. Let y be the $\sum_{i=1}^{r-1} c_i + l$ th element from the end in this snake. Then $y = m+1 - \sum_{i=1}^{r-1} c_i + l = m+1-x$. Since $\sum_{i=1}^{r-1} c_i + l \leq \sum_{i=1}^r c_i$, y will be lying in one of the columns $k, \dots, k-r+1$.

Having the above mapping, we know there is also a one-to-one mapping in $\{\cup_{t=1}^r T_t\} \rightarrow \{\cup_{t=k-r+1}^k T_t\}$ and also in $\{\cup_{t=1}^{r-1} T_t\} \rightarrow \{\cup_{t=k-r}^k T_t\}$. Hence, there exists $\mathcal{F} : T_r \rightarrow T_{k+1-r}$ satisfying the lemma's conditions.

□

Proof of Theorem 25, part 3. Considering any $m \times k$ tuple cylinder $TS(\mathcal{O})$, Lemma 44 shows that $TS(\mathcal{O})$ is totally covered by k snakes. Therefore, each element $1 \leq i \leq m$ appears k times in the cylinder and therefore the average of f_i as defined in Theorem 25 part 3 is independent of \mathcal{O} and equal to k/m .

Lemma 50 shows that the number of j s in any column i is equal to the number of $m-j+1$'s in column $k-i+1$ of $TS(\mathcal{O})$. Thus, $\sum_{x \in \mathcal{O}} g_{i,j}(x) = \sum_{x \in \mathcal{O}} g_{k-i+1, m-j+1}(x)$. In other words, $\forall 1 \leq i \leq k, 1 \leq j \leq m, g_{i,j} - g_{k-i+1, m-j+1}$ is 0-mesic in W -orbits of $S_{k,m}$. □

3.2 Proof of Theorem 29

In this subsection we will prove Theorem 29. Remember the definitions of tuple board, snake, snake map and the correspondence to the action of winching with lower bounds.

Definition 51. For the set of lower bounds $l = (l_1, \dots, l_k)$ and $i \in k$, let \bar{W}_i be defined as: $\bar{W}_i : S_{k,m} \rightarrow S_{k,m}$; $\forall x \in S_{k,m} \bar{W}_i(x) = \max\{l_i, \bar{W}_i(x)\}$, where \bar{W}_i is defined as in Definition 40.

Note that $\forall x \in S_{k,m}, \bar{W}_i \circ W_i(x) = x$. We call \bar{W}_i **inverse winching** at index i .

Definition 52. For arbitrary k and ν a permutation of $[k]$, $\bar{W}_\nu : S_{k,m} \rightarrow S_{k,m}$ is defined by $\bar{W}_\nu = \bar{W}_{\nu(1)} \circ \bar{W}_{\nu(2)} \circ \dots \circ \bar{W}_{\nu(k)}$ and we have $\forall x \in S_{k,m}, \bar{W}_\nu(W_\nu(x)) = x$.

In contrast to what we showed in Lemma 42 for W_ν -snakes, \bar{W}_ν -snakes do not necessarily cover all the numbers $1, 2, \dots, m$. As depicted in Figure 3, they only contain l_i, \dots, m where l_i is one of the lower bounds. The following lemma states this formally:

Lemma 53. Consider a tuple cylinder $TS(\mathcal{O})$ which is constructed by applying the action of \bar{W}_ν to an arbitrary $x \in S_{k,m}$. Let the lower bounds for this action be $l = (l_1, l_2, \dots, l_k)$. Any snake in this tuple board starts in some column q and with s_{l_q} , and ends in the last column of the tuple cylinder with s_m .

Proof. Consider $x \in \mathcal{O}$ and snake s in the tuple board $T = TB(x) = [x^1, \dots, x^n]$. We assume that $s = (s_f \dots s_t)$. Following the definitions of winching and snakes, we can see that unless $t = m$ we can

append more cells to the tail of the snake, and if $l_i < f$ we can append more cells to the head of the snake.

□

Proof of Theorem 29.

Any tuple cylinder corresponding to action of \underline{W}_ν can be partitioned to snakes that start with some s_{l_q} and end in s_m . Therefore, if f is a function that have the same average on all the numbers contained in any snake, it will have the same average over all the elements in the tuple cylinders. Therefore, we will have the result.

□

3.3 Proof of Theorem 33

In this section, we will prove Theorem 33. Remember the definitions of tuple board, tuple cylinder, snake, snake map. Consider $x \in S_n$ and the action of WZ_ν for some arbitrary permutation ν of $[n]$. Note that in the case of winching with zeros, since we might have a bunch of zeros in our tuple board the snake does not necessarily start in column 1. However, it is a consecutive collection of numbers $1, 2, \dots, n$ as it was the case in Lemma 42.

Definition 54. Let M_n be the set of all sequences (c_1, \dots, c_n) that have k preceding 0s for some $0 \leq k \leq n-1$, $c_{k+1} \dots c_n > 0$, and $\sum_{i=k+1}^n c_i = n$. Let M_n be the set of all possible snake maps. We define the action **crawl** $C : M_n \rightarrow M_n$ such that for any $c \in M_n$, $C(c) = c'$ where,

For $1 \leq i \leq n-1$,

$$c'_i = \begin{cases} \max\{0, c_{i+1}-1\} & \text{If } c_1, \dots, c_i \leq 1; \\ c_{i+1} & \text{otherwise.} \end{cases}$$

And, $c'_n = n - \sum_{i=1}^{n-1} c'_i$.

Lemma 55. Consider some arbitrary ν a permutation of $[n]$. Let $T_{m \times n}$ be a tuple board corresponding to WZ_ν , i a row in T containing a 1, and snake s starting at row i . Let j be the smallest number greater than i containing another 1, and s' the snake starting at row j .

1. We have $j = i + 2$.
2. Let $c = (c_1, \dots, c_n)$ be snake map of s and $c' = (c'_1, \dots, c'_n)$ be snake map of s' . We have $c' = C(c)$

3. Any element of the tuple board is either a 0 or it belongs to a snake.

Proof. For simplicity we assume $\nu = (1, 2, \dots, n)$, and drop subscript from W . The proof any arbitrary permutation ν of $[n]$ will be similar.

Proof of Part 1. First, we argue that $c_1 \leq 1$. We know that $\sum_{i=1}^n c_i = n$. If $c_1 \neq 0$ then we have

$$c_2, \dots, c_n > 0 \Rightarrow c_2 + \dots + c_n \geq n-1 \Rightarrow c_1 \leq n - (n-1) = 1.$$

Since there is a 1 in row i if $T(i, 1) = 0$, $T(i, 2) = 0$ or $T(i, 2) = 1$. In both cases, applying winching will derive, $T(i+1, 1) = 0$. If $T(i, 1) = 1$ since $c_1 \leq 1$ we have $T(i, 2) = 2$, and $T(i+1, 1) = 0$. Thus, in both cases $T(i+1, 1) = 0$. Consider the first column where s turns down and let it be column j . We have, $c_1, \dots, c_{l-1} \leq 1$, and $T(i, l) + 1 = T(i+1, l)$. Note that for any $1 \leq k \leq l$, $T(i, k) = 0$ or $T(i, k) = T(i, k-1) + 1$. Hence, applying WZ to $y = T(i, \cdot)$ will result in $T(i+1, k) = 0$ for $1 \leq k \leq l-1$, and $T(i+1, l) > 1$. Since the rest of entries in row $i+1$ will be in increasing order, there will not be any 1 in this row. In the $i+2$ nd row, we will have $T(i+2, k) = 0$ for $1 \leq k \leq l-2$, and $T(i+2, l-1) = 1$. As a result, $j = i+2$.

Proof of Part 2. As we argued in part 1, s' starts at $T(i+2, l-1)$. Since elements of s and s' will be nonzero from this point to the right then by the same argument presented for regular winching in the proof of Lemma 44) we can show the snakes will move in parallel to each other. Note that in row $i+1$ we have zeros until we get to $T(i+1, l)$. In row $i+2$ we have zeros in $T(i+2, 1), \dots, T(i+2, l-2)$, i.e. $c'_1 = \dots = c'_{l-2} = 0$. At column $l-1$, s' will start and move parallel to s but there is a zero between s and s' in column $l-1$. Therefore, $c'_{l-1} = c_l - 1$, and there is no zero between the remaining part of s and s' , thus $c'_k = c_{k+1}$, for $l \leq k \leq n-1$. The rest of s' will continue in column n .

Proof of Part 3. It is clear from the above argument that the space between the initial segments of any two snakes is filled with zeros. □

Lemma 56.

For any $c \in M_n$ we have, $C^n(c) = c$.

We will need the following definitions and lemmas to prove Lemma 56.

Definition 57. Consider the set M_n . We define the one-to-one map $\mathcal{F} : M_n \rightarrow \{0, 1\}^{2n}$ as follows:

for all $c \in M_n$, $\mathcal{F}(c) = b = (b_1, \dots, b_{2n})$ where for $1 \leq i \leq n$,

$$b_i = \begin{cases} 1 & \text{If } \exists k; c_1 + \dots + c_k = i; \\ 0 & \text{Otherwise.} \end{cases}$$

And, for $n < i \leq 2n$, $b_i = \neg b_{i-n+1}$.

Lemma 58. \mathcal{F} is one-to-one.

Proof. Assume $\mathcal{F}(x) = \mathcal{F}(y) = w$, and let j be the smallest index where $w_j = 1$. We have $x_1 = y_1 = j$. The next nonzero index will determine that $x_2 = y_2$ and likewise, we can verify that all entries of x and y are equal. \square

Definition 59. Let $\mathcal{B}_n \subset \{0, 1\}^{2n}$ be the set of all $b \in \{0, 1\}^{2n}$ such that for all, $1 \leq i \leq n$, $b_i = \neg b_{n+i}$. We define the action of rotation $\mathcal{R} : \mathcal{B} \rightarrow \mathcal{B}$ on this set as follows: Partition b into maximal blocks of $1^k 0$, remove the leftmost block, and put it on the right.

Example 60. Let $b = (110010001101)$. The partitioning of b will be $(110.0.10.0.0.110.1)$. Therefore, $\mathcal{R}(b) = (0.10.0.0.110.1.110)$.

Lemma 61. For $c \in M_n$, we have $\mathcal{F}(C(c)) = \mathcal{R}(\mathcal{F}(c))$

Proof. Consider an arbitrary $c = (c_1, \dots, c_n) \in M_n$. Let's say we have $c_1 = \dots = c_{k-1} = 0$, and c_k is the leftmost nonzero element in c . Consider the set $\mathcal{A} = \{a_1 = c_k, a_2 = c_k + c_{k+1}, \dots, a_{n-k} = \sum_{i=k}^n c_i\}$. Let $C(c) = c'$ and $b = (b_1, \dots, b_n)$, the binary word representing \mathcal{A} . In other words for all $a, a \in \mathcal{A} \Leftrightarrow b_a = 1$.

Similarly, let $\mathcal{A}' = \{c_{k'}, c_{k'} + c_{k'+1}, \dots, \sum_{i=k'}^n c'_i\}$ where k' is the leftmost nonzero element in c' and b' be \mathcal{A}' 's binary representation.

According to definition of crawl we know that, if $c_k = \dots = c_{l-1} = 1$, we will have $c'_{k-1}, \dots, c'_{l-2} = 0$ and $c'_{l-1} = c_l - 1$, where l is the leftmost element greater than 1. This means that if we have $a_1 = 1, a_2 = 2, a_3 = 3, \dots, a_{l-1} = l$, they should be removed from \mathcal{A} to make \mathcal{A}' . In other words, any set of consecutive elements starting from a 1 will be removed in \mathcal{A}' . Moreover, c_l will be decremented which means a_1 and the rest of the elements in \mathcal{A} will be decreased by l except the last one which should always be an n . Now, let's see how b will change accordingly. We remove consecutive elements starting with a 1 from \mathcal{A} which means we remove the preceding 1s from b until we hit a 0. All the other elements will be decreased by l which means they should be shifted to left by l positions. This is equivalent to removing the first block from b . Now, we need to add $b'_{n-l+1}, \dots, b'_{n-1} = 0$. And $b'_n = 1$ because c'_n should be increased by l to make the length of the snake equal to n . This whole process is removing the leftmost block and adding its negation to the right, which is equivalent to a rotation of a block in $\mathcal{F}(c)$. \square

Lemma 62. $\forall x \in \mathcal{B}_n, \quad \mathcal{R}^n(x) = x$.

Proof. Consider any arbitrary x , any block in x has a single 0. Moreover, the number of zeros in x is n . Therefore, after n rotations x will get back to its initial state. \square

Proof of Lemma 56. From Lemma 62 and 61 and the fact that \mathcal{F} is a one-to-one function we have, $\forall c \in \mathcal{M}_n, \quad C^n(c) = n$.

Proof of Theorem 33 Part 1. By employing Lemma 55 we can verify that the snakes appear in alternating rows. By Lemma 56 we know that each snake gets back to itself after n crawls. Thus, $T(1, \cdot) = T(2n+1, \cdot)$ where T is a tuple board corresponding to WZ , and $WZ^{2n}(x) = x$. \square

Proof of Theorem 33 Part 2. Part 2. Using Lemma 55 part 3 we know that half of any tuple board is filled with zeros, and the rest is filled by equal repetitions of numbers 1 to n . In addition, there are n snakes in any tuple board and in any snake j appears once and only once. Therefore, each element will appear n times in the tuple board and the average of $f_j = 1/2$ for each j . \square

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